The University of Chicago

THEORY OF SURFACES IN CONJUGATE PARAMETERS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
DECEMBER, 1940

By
JEROME MICHAEL SACHS

CHICAGO, ILLINOIS
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THEORY OF SURFACES IN CONJUGATE PARAMETERS

1. Introduction. This thesis is concerned with the investigation of certain portions of the projective differential geometry of a conjugate net on an analytic surface in ordinary space, and with the geometry of the surface itself expressed in terms of the conjugate parameters. Following some of the basic theory of conjugate nets, the power series representing the surface will be extended to terms of the fifth degree. The expansions for the parametric curves, which will be useful in subsequent sections, will also be presented.

Next, the relationship between the section of the surface by the osculating plane of one of the parametric curves and the projection of that curve onto the osculating plane will be investigated, and the conditions for various orders of contact will be obtained. We shall then consider the projection of the curves of the net onto the tangent plane, and their relations to the projections of sections of the surface by various associated planes. We shall find the loci of certain linear osculants of sections of the surface by a variable plane through the axis.

In the fourth section a pair of quadric cones will be obtained as analogues of the Quadrics of Lie. Then the equations for the osculating quadric cones of the parametric curves will be developed, and these will be compared with the cones analogous to the Quadrics of Lie.

The final section deals with the ruled surfaces of rays for the parametric curves of the net. We shall study the osculating

quadrics of these ruled surfaces; and discuss the properties of their curve of intersection, as well as the relationship between the polar planes of certain points with respect to these quadrics.

2. Analytic basis. Power series expansions. We shall first present some of the basic theory of a surface referred to a conjugate net, and then develop a power series representation for the surface in conjugate parameters. We consider an analytic surface S in ordinary space having the vector parametric equation $x = x(u,v)$. The parametric curves on S form a conjugate net if and only if the coordinates x satisfy an equation of the form,

$$x_{uv} = cx + ax_u + bx_v.$$

The points P_{x_1} and $P_{x_{-1}}$ defined by the equations

$$x_1 = x_v - ax, \qquad x_{-1} = x_u - bx \tag{1}$$

are known as the first and minus-first Laplace transform points. In the notation of Lane[1], the system of differential equations upon which the theory of conjugate nets is based can be written in the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0), \end{aligned} \tag{2}$$

in which the point y lies on the line of intersection of the osculating planes of the parametric curves through P_x, and is the harmonic conjugate of P_x with respect to the focal points of this

---

[1] E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago: University of Chicago Press, 1932, p. 138.

line. The line of intersection of the two osculating planes is known as the axis at P_x.

The first partial derivatives of y are given by the equations

$$(3)\qquad \begin{aligned} y_u &= fx + mx_u + sx_v + Ay = Px + mx_{-1} + sx_1 + Ay, \\ y_v &= gx + tx_u - mx_v + By = Qx + tx_{-1} - mx_1 + By, \end{aligned}$$

where

$$(4)\qquad \begin{aligned} fN &= c_v + ac + bq - c\delta - q_u, & gL &= c_u + bc + ap - c\alpha - p_v, \\ mN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v, \\ sN &= b_v + ab + c - \delta_u, & -mL &= b_u + b^2 - b\alpha - p, \\ A &= b - N_u/N, & B &= a - L_v/L, \end{aligned}$$

and the invariants P and Q of Davis[2] are given by the equations

$$(5)\qquad P = f + bm + sa, \qquad Q = g - am + bt.$$

Some other invariants of the net are given by

$$(6)\qquad \begin{aligned} 8\mathcal{B}' &= 4a - 2\delta + r_v/r, & 8\mathcal{C}' &= 4b - 2\alpha - r_u/r, \\ H &= c + ab - a_u, & K &= c + ab - b_v, \\ r &= N/L, & \mathcal{D} &= 2Lm, \\ \mathcal{H} &= sN, & \mathcal{K} &= tL. \end{aligned}$$

We shall use as our local tetrahedron of reference the covariant tetrahedron with vertices at the points x, x_{-1}, x_1, y, so that a point X whose coordinates are given by

$$X = y_1x + y_2x_{-1} + y_3x_1 + y_4y$$

shall have local coordinates proportional to y_1, y_2, y_3, y_4.

---

[2] W. M. Davis, Contributions to the Theory of Conjugate Nets, doctoral dissertation, University of Chicago, 1933, p. 5.

The coordinates of a point X near P_x on the surface can be represented by means of the Taylor's series

$$X = x + x_u \Delta u + x_v \Delta v + x_{uu} \Delta u^2/2 + x_{uv} \Delta u \Delta v + \cdots .$$

By means of (1), (2), and (3) it is possible to express all the derivatives of x as linear combinations of x, x_{-1}, x_1, y. Substituting the derivatives in the above Taylor series and collecting coefficients of x, x_{-1}, x_1, y we have the local coordinates of the point X expressed in the form

$$\begin{aligned} y_1 = {} & 1 + b\Delta u + a\Delta v + (p + \alpha b)\Delta u^2/2 + (c + 2ab)\Delta u \Delta v \\ & + (q + \delta a)\Delta v^2/2 + [(p + \alpha b)_u + b(p + \alpha b) + \alpha Lm \\ & + LP]\Delta u^3/6 + [(c + 2ab)_u + b(c + 2ab) + aLm + bH]\Delta u^2 \Delta v/2 \\ & + [(c + 2ab)_v + a(c + 2ab) - bNm + aK]\Delta u \Delta v^2/2 \\ & + [(q + \delta a)_v + a(q + \delta a) - Nm\delta + NQ]\Delta v^3/6 + \cdots , \end{aligned}$$

$$\begin{aligned} y_2 = {} & \Delta u + \alpha \Delta u^2/2 + a\Delta u \Delta v + (p + \alpha_u + \alpha^2 + Lm)\Delta u^3/6 \\ & + (\alpha_v + a\alpha + Lt)\Delta u^2 \Delta v/2 + (a_v + a^2)\Delta u \Delta v^2/2 \\ & + Nt\Delta v^3/6 + [(p + \alpha b)_u + b(p + \alpha b) + \alpha Lm + LP \\ & + (p + \alpha^2 + \alpha_u + Lm)_u + (\alpha - b)(p + \alpha^2 + \alpha_u + Lm) \\ & + Lm(b + \alpha - \gamma_u)]\Delta u^4/24 + [(p + \alpha^2 + \alpha_u + Lm)_v \\ & + a(p + \alpha^2 + \alpha_u + Lm) + Lt(b + \alpha - \gamma_u)]\Delta u^3 \Delta v/6 \\ & + [(\alpha_v + a\alpha + Lt)_v + a(\alpha_v + a\alpha + 2Lt)]\Delta u^2 \Delta v^2/4 \\ & + [(q + \delta a)_v + a(q + \delta a) + NQ - \delta Nm + (Nt)_u \\ & + Nt(\alpha - b) + Nm(a + \delta + \gamma_v)]\Delta u \Delta v^3 + [(Nt)_v \\ & + Nt(2a + \delta + \gamma_v)]\Delta v^4/24 + \cdots , \end{aligned}$$

$$\begin{aligned} y_3 = {} & \Delta v + b\Delta u \Delta v + \delta \Delta v^2/2 + Ls\,\Delta u^3/6 + (p + \alpha b \\ & - Lm)\Delta u^2 \Delta v/2 + (\delta_u + \delta b + Ns)\Delta u \Delta v^2/2 + (q + \delta_v \\ & + \delta^2 - Nm)\Delta v^3/6 + [(Ls)_u + Ls(2b + \alpha - \gamma_u)]\Delta u^4/24 \\ & + [(p + \alpha b)_u + b(p + \alpha b) + \alpha Lm + LP + (Ls)_v \end{aligned}$$

$$+ Ls(\delta - a) - Lm(b + \alpha - \chi_u)]\Delta u^3\Delta v/6 + [(p + \alpha b)_v + a(p + \alpha b) + \alpha K + LQ + (p + \alpha b - Lm)_v + (p + \alpha b - Lm)(\delta - a) - Lma]\Delta u^2\Delta v^2/4 + [(q + \delta_v + \delta^2 - Nm)_u + b(q + \delta_v + \delta^2 - Nm) + Ns(a + \delta + \chi_v)]\Delta u\,\Delta v^3/6 + [(\delta_v + q + \delta^2 - Nm)N + \{N(a + \delta + \chi_v)\}_v + N(a + \delta + \chi_v)B]\Delta v^4/24 + \cdots ,$$

$$y_4 = L\Delta u^2/2 + N\Delta v^2/2 + L(b + \alpha - \chi_u)\Delta u^3/6 + La\,\Delta u^2\Delta v/2 + Nb\,\Delta u\,\Delta v^2/2 + N(a + \delta + \chi_v)\Delta v^3/6 + \{L(p + \alpha_u + \alpha^2 + Lm) + [L(b + \alpha - \chi_u)]_u + LA(b + \alpha - \chi_u)\}\Delta u^4/24 + \{LsN + [L(b + \alpha - \chi_u)]_v + LB(b + \alpha - \chi_u)\}\Delta u^3\Delta v/6 + \{N(p + \alpha b - Lm) + (aL)_v + LBa\}\Delta u^2\Delta v^2/4 + \{NtL + [N(a + \delta + \chi_v)]_u + NA(a + \delta + \chi_v)\}\Delta u\,\Delta v^3/6 + \{N(q + \delta_v + \delta^2 - Nm) + [N(a + \delta + \chi_v)]_v + NB(a + \delta + \chi_v)\}\Delta v^4/24 + \{L[(p + \alpha b)_u + b(p + \alpha b) + \alpha Lm + LP] + L(p + \alpha_u + \alpha^2 + Lm)_u + L(p + \alpha_u + Lm)(\alpha - b) + L^2m(b + \alpha - \chi_u) + [L(p + \alpha_u + \alpha^2 + Lm) + \{L(b + \alpha - \chi_u)\}_u + LA(b + \alpha - \chi_u)]_u + A[L(p + \alpha_u + \alpha^2 + Lm) + \{L(b + \alpha - \chi_u)\}_u + LA(b + \alpha - \chi_u)]\}\Delta u^5/120 + \{N[(Ls)_u + Ls(2b + \alpha - \chi_u)] + [L(p + \alpha_u + \alpha^2 + Lm) + \{L(b + \alpha - \chi_u)\}_u + LA(b + \alpha - \chi_u)]_v + B[L(p + \alpha_u + \alpha^2 + Lm) + \{L(b + \alpha - \chi_u)\}_u + LA(b + \alpha - \chi_u)]\}\Delta u^4\Delta v/24 + \{[LsN + \{L(b + \alpha - \chi_u)\}_v + LB(b + \alpha - \chi_u)]_v + B[LsN + \{L(b + \alpha - \chi_u)\}_v + LB(b + \alpha - \chi_u)] + N[(p + \alpha b)_u + b(p + \alpha b) + \alpha Lm + LP + (Ls)_v + Ls(\delta - a) - Lm(b + \alpha - \chi_u)]\}\Delta u^3\Delta v^2/12 + \{[NtL + \{N(a + \delta + \chi_v)\}_u + NA(a + \delta + \chi_v)]_u + A[NtL + \{N(a + \delta + \chi_v)\}_u + NA(a + \delta + \chi_v)] + L[(q + \delta a)_v + a(q + \delta a) - \delta Nm + NQ + (Nt)_u + Nt(\alpha - b) + Nm(a + \delta + \chi_v)]\}\Delta u^2\Delta v^3/12 + \{[(Nt)_v + Nt(2a \tag{7}$$

$$+ \delta + \chi_v)]L + [N(q + \delta_v + \delta^2 - Nm) + \{N(a + \delta + \chi_v)\}_v + NB(a + \delta + \chi_v)]_u + A[N(q + \delta_v + \delta^2 - Nm) + \{N(a + \delta + \chi_v)\}_v + NB(a + \delta + \chi_v)]\} \Delta u\, \Delta v^4/24 + \{N[(q + \delta a)_v + a(q + \delta a) - \delta Nm + NQ] + N(q + \delta_v + \delta^2 - Nm)_v + N(q + \delta_v + \delta^2 - Nm)(\delta - a) - N^2 m(a + \delta + \chi_v) + [N(q + \delta_v + \delta^2 - Nm) + \{N(a + \delta + \chi_v)\}_v + NB(a + \delta + \chi_v)]_v + B[N(q + \delta_v + \delta^2 - Nm) + \{N(a + \delta + \chi_v)\}_v + NB(a + \delta + \chi_v)]\} \Delta v^5/120 + \cdots .$$

Defining A by placing

$$A = b\,\Delta u + a\,\Delta v + (p + \alpha b)\,\Delta u^2/2 + \cdots ,$$

and then introducing nonhomogeneous coordinates by means of the substitutions

$$y_1 = 1 + A,$$
$$x = y_2/y_1 = y_2(1 - A + A^2 - A^3 + \cdots),$$
$$y = y_3/y_1 = y_3(1 - A + A^2 - A^3 + \cdots),$$
$$z = y_4/y_1 = y_4(1 - A + A^2 - A^3 + \cdots),$$

we obtain the following expansions:

$$x = \Delta u - (8\mathfrak{C}' + \chi_u)\,\Delta u^2/4 + \{2(4\mathfrak{C}' + \chi_u/2)^2 - 2(4\mathfrak{C}' + \chi_u/2)_u - \mathfrak{D}\}\Delta u^3/12 - H\,\Delta u^2\,\Delta v/2 + Nm\,\Delta u\,\Delta v^2/2 + Nt\,\Delta v^3/6 + \{3[(4\mathfrak{C}' + \chi_u/2)^2]_u - 2(4\mathfrak{C}' + \chi_u/2)_{uu} - 2(4\mathfrak{C}' + \chi_u/2)^3 - \mathfrak{D}_u - 6LP + 6\mathfrak{D}(4\mathfrak{C}' + \chi_u/3)\}\Delta u^4/48 + \{2[(4\mathfrak{C}' + \chi_u/2)^2]_v - 2(4\mathfrak{C}' + \chi_u/2)_{uv} + 12K(4\mathfrak{C}' + \chi_u/2) - 2\mathcal{K}(4\mathfrak{C}' + 3\chi_u/2) - \mathfrak{D}_v - 6LQ\}\Delta u^3\,\Delta v/12 + \{4H(4\mathfrak{B}' - \chi_v/2) - 2(4\mathfrak{C}' + \chi_u/2)_{vv} + 2\mathcal{K}_v - r\mathfrak{D}(4\mathfrak{C}' + \chi_u/2) - 4NP + 4(H - K)_v + 2(r\mathfrak{D})_u\}\Delta u^2\,\Delta v^2/8 + \{2(r\mathcal{K})_u - 2r\mathcal{K}(4\mathfrak{C}' + \chi_u/2) - r\mathfrak{D}(4\mathfrak{B}' - 3\chi_v/2)\}\Delta u\,\Delta v^3/12 + \{(r\mathcal{K})_v - r\mathcal{K}(4\mathfrak{B}' - 3\chi_v/2)\}\Delta v^4/24 + \cdots ,$$

$$
\begin{aligned}
y = {} & \Delta v - (8\mathfrak{B}' - \chi_v)\Delta v^2/4 + Ls\,\Delta u^3/6 - Lm\,\Delta u^2\Delta v/2 \\
& - K\Delta u\,\Delta v^2/2 + \left\{ 2(4\mathfrak{B}' - \chi_v/2)^2 - 2(4\mathfrak{B}' - \chi_v/2)_v \right. \\
& \left. + \mathfrak{D}r \right\}\Delta v^3/12 + \left\{ (\mathfrak{A}/r)_u - (\mathfrak{A}/r)(4\mathfrak{C}' + 3\chi_u/2) \right\}\Delta u^4/24 \\
& + \left\{ 2(\mathfrak{A}/r)_v - 2\mathfrak{A}/r\,(4\mathfrak{B}' - \chi_v/2) + \mathfrak{D}(4\mathfrak{C}' + 3\chi_u/2) \right\} \\
& \cdot\Delta u^3\Delta v/12 + \left\{ 4K(4\mathfrak{C}' + \chi_u/2) - 2(4\mathfrak{B}' - \chi_v/2)_{uu} \right. \\
& + 2\mathfrak{A}_u + \mathfrak{D}(4\mathfrak{B}' - \chi_v/2) - 4LQ + 4(K - H)_v - \left. 2\mathfrak{D}_v \right\}\Delta u^2\Delta v^2/8 \\
& + \left\{ 2[(4\mathfrak{B}' - \chi_v/2)^2]_u - 2(4\mathfrak{B}' - \chi_v/2)_{uv} + 12H(4\mathfrak{B}' \right. \\
& - \chi_v/2) - 2\mathfrak{A}(4\mathfrak{B}' - 3\chi_v/2) + (r\mathfrak{D})_u - \left. 6NP \right\}\Delta u\,\Delta v^3/12 \\
& + \left\{ 3[4\mathfrak{B}' - \chi_v/2)^2]_v - 2(4\mathfrak{B}' - \chi_v/2)_{vv} - 2(4\mathfrak{B}' - \chi_v/2)^3 \right. \\
& \left. + r\mathfrak{D}_v - 6NQ - 6r\mathfrak{D}(4\mathfrak{B}' - \chi_v/3) \right\}\Delta v^4/48 + \cdots ,
\end{aligned}
$$

(8)

$$
\begin{aligned}
z = {} & L\Delta u^2/2 + N\Delta v^2/2 - [L(4\mathfrak{C}' + 3\chi_u/2)]\Delta u^3/6 - [N(4\mathfrak{B}' \\
& - 3\chi_v/2)]\Delta v^3/6 + \left\{ L(4\mathfrak{C}' + \chi_u/2)^2 - 2L(4\mathfrak{C}' + \chi_u)_u \right. \\
& + L\chi_u(4\mathfrak{C}' + 3\chi_u/2) - \left. 2L\mathfrak{D} \right\}\Delta u^4/24 + \left\{ L[\mathfrak{A} - 3K \right. \\
& \left. - (4\mathfrak{C}' + 3\chi_u/2)_v] \right\}\Delta u^3\Delta v/6 + \left\{ N[\mathfrak{K} - 3H - (4\mathfrak{B}' \right. \\
& \left. - 3\chi_v/2)_u] \right\}\Delta u\,\Delta v^3/6 + \left\{ N(4\mathfrak{B}' - r_v/2r)^2 - 2N(4\mathfrak{B}' - \chi_v)_v \right. \\
& - N\chi_v(4\mathfrak{B}' - 3\chi_v/2) + \left. 2Nr\mathfrak{D} \right\}\Delta v^4/24 + \left\{ 2L(4\mathfrak{C}' \right. \\
& + 3\chi_u/2)[\chi_{uu} - \chi_u^2 + 8\mathfrak{D} - (4\mathfrak{C}' + \chi_u/2)^2] - 6L(4\mathfrak{C}' \\
& + 5\chi_u/6)_{uu} - 18L^2P + 5L[(4\mathfrak{C}' + \chi_u/2)^2]_u + 6L\chi_u(4\mathfrak{C}' \\
& + 7\chi_u/6)_u + 4L\mathfrak{D}_u - \left. 9L(r\mathfrak{D})_u/r \right\}\Delta u^5/240 + L\left\{ r(\mathfrak{A}/r)_u \right. \\
& - 6K_u + [4\mathfrak{C}' + \chi_u/2)^2 + \mathfrak{D} + \chi_u(4\mathfrak{C}' + 3\chi_u/2) - 2(4\mathfrak{C}' \\
& + \chi_u)_u]_v + (4\mathfrak{C}' + 3\chi_u/2)(4K - \mathfrak{A}) \left. \right\}\Delta u^4\,\Delta v/24 + \left\{ L\mathfrak{A}_v \right. \\
& + N(\mathfrak{A}/r)_v - L(4\mathfrak{C}' + 3\chi_u/2)_{vv} - L\mathfrak{A}(4\mathfrak{B}' - \chi_v/2) \\
& - \left. 3LK_v \right\}\Delta u^3\Delta v^2/12 + \left\{ N\mathfrak{K}_u + L(r\mathfrak{K})_u - N(4\mathfrak{B}' - 3\chi_v/2)_{uu} \right. \\
& - N\mathfrak{K}(4\mathfrak{C}' + \chi_u/2) - \left. 3NH_u \right\}\Delta u^2\,\Delta v^3/12 + N\left\{ (r\mathfrak{K})_v/r \right. \\
& - 6H_v + [(4\mathfrak{B}' - \chi_v/2)^2 - r\mathfrak{D} - \chi_v(4\mathfrak{B}' - 3\chi_v/2) - 2(4\mathfrak{B}' \\
& - \chi_v)_v]_u + (4\mathfrak{B}' - 3\chi_v/2)(4H - \mathfrak{K}) \left. \right\}\Delta u\,\Delta v^4/24 \\
& + \left\{ 2N(4\mathfrak{B}' - 3\chi_v/2)[\chi_{vv} - \chi_v^2 - 8r\mathfrak{D} - (4\mathfrak{B}' - \chi_v/2)^2] \right. \\
& - 6N(4\mathfrak{B}' - 5\chi_v/6)_{vv} - 18N^2Q + 5N[(4\mathfrak{B}' - \chi_v/2)^2]_v \\
& - \left. 6N\chi_v(4\mathfrak{B}' - 7\chi_v/6)_v - 4N(r\mathfrak{D})_v + 9Nr\mathfrak{D}_v \right\}\Delta v^5/240 + \cdots .
\end{aligned}
$$

In the computation involved in finding the coefficients in the foregoing expansions, the reduction to invariant form is accomplished by judicious use of the equations (4), (5), and (6). In particular, substitutions and grouping methods which led to combinations of $(2b - \alpha)$ and $(2a - \delta)$ were most effective. Frequent use was also made of the substitutions

$$p + \alpha b = b_u + b^2 + Lm, \qquad q + \delta a = a_v + a^2 - Nm.$$

By the method of undetermined coefficients, and using the expansions (8), we get an expansion for z in terms of x and y. This expansion has been carried out to terms of the fourth degree by Lane.[3] To terms of the fifth degree it is

$$\begin{aligned} z = {} & (Lx^2 + Ny^2)/2 + 4(L\mathfrak{C}'x^3 + N\mathfrak{B}'y^3)/3 + e_{40}x^4 + e_{31}x^3y \\ & + e_{13}xy^3 + e_{04}y^4 + e_{50}x^5 + e_{41}xy^4 + e_{32}x^3y^2 + e_{23}x^2y^3 \\ & + e_{14}xy^4 + e_{05}y^5 + \cdots , \end{aligned} \tag{9}$$

where

$$\begin{aligned} e_{40} &= L\mathfrak{C}'[12\mathfrak{C}' + (\log \mathfrak{C}'\sqrt{r})_u]/3, \\ e_{31} &= L(H - \mathcal{H})/6, \\ e_{50} &= L\{2\mathfrak{C}'(16\mathfrak{C}' + \chi_u)(24\mathfrak{C}' + \chi_u) + 6\mathfrak{C}'(20\mathfrak{C}' + \\ &\quad + \chi_u)(\log \mathfrak{C}'\sqrt[3]{20\mathfrak{C}' + \chi_u})_u + 4\mathfrak{C}'_{uu} + 3LP - 4\mathfrak{C}'\mathfrak{D}\}/60, \\ e_{41} &= L\{[\chi_u(4\mathfrak{C}' + 3\chi_u/2) + 8\mathfrak{C}'_u - 3(4\mathfrak{C}' + \chi_u/2)^2 - 3\mathfrak{D}]_v \\ &\quad + 6K_u + 48H\mathfrak{C}' + 2(4\mathfrak{C}' + \chi_u)(2\mathcal{H} - 4K - 3\mathcal{H}) \\ &\quad + 4\chi_u(K + \mathcal{H})\}/24, \\ e_{32} &= L\{[8\mathfrak{C}'_v + 3(K - \mathcal{K})]_v + (H - \mathcal{H})(4\mathfrak{B}' - \chi_v/2) \\ &\quad - 2r\mathfrak{C}'\mathfrak{D}\}/12. \end{aligned}$$

---

[3] E. P. Lane, A Canonical Power Series Expansion for a Surface, Transactions of the American Mathematical Society, vol. 37 (1935), p. 463.

In the preceding formulas we have placed $l = \log r$. The coefficients not written can be obtained from those written by means of symmetrical substitutions.

The expansions for the parametric curves of the conjugate net will be used several times in the following sections. We present here the expansions for the u-curve, those for the v-curve being symmetrical. For the u-curve Lane and MacQueen obtained the expressions[4]

$$
\begin{aligned}
y &= \mathscr{A}x^3/6r + \mathscr{A}[I + 16\mathscr{C}']x^4/24r + \mathscr{A}a_5x^5/120r \\
&\qquad + \mathscr{A}a_6x^6/720r + \cdots, \\
z &= Lx^2/2 + 4L\mathscr{C}'x^3/3 + Lb_4x^4/24 + Lb_5x^5/120 + Lb_6x^6/720 + \cdots,
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
b_4 &= 4(2\mathscr{C}'_u + \mathscr{C}'\mathscr{I}_u + 24\mathscr{C}^2), \\
b_5 &= b_{4u} + (16\mathscr{C}' + \mathscr{I}_u)b_4 - 8\mathscr{C}'\mathscr{D} + 6LP, \\
a_5 &= 3b_4 + 24\mathscr{C}'I + 3I^2/2 + J, \\
a_6 &= 4b_5 + 6Ib_4 + E + J_u + J\mathscr{I}_u - I\mathscr{D}/2 + 2LP, \\
I &= (\log\mathscr{A})_u + 4\mathscr{C}' + \mathscr{I}_u/2, \\
J &= I_u - I^2/2 + I\mathscr{I}_u/2 + 4\mathscr{C}'I + \mathscr{D}, \\
E &= 3I^3 + 4IJ + 40\mathscr{C}'J + 48\mathscr{C}'I^2.
\end{aligned}
$$

3. Plane sections and projections. We shall now compare the curve of intersection of the surface and the osculating plane of the parametric u-curve C_u at P_x with the projection of C_u onto that osculating plane from the ray point of C_u at P_x. The equations of the section of the surface by the osculating plane of C_u at P_x, namely, the plane $y = 0$, are $y = 0$ and equation (9). The equation obtained by setting y equal to zero in (9) is

[4] E. P. Lane, and M. L. Mac Queen, Curves of a Conjugate Net, Duke Mathematical Journal, vol. 5 (1939), p. 695.

$$(12) \qquad z = Lx^2/2 + 4L\mathfrak{C}'x^3/3 + e_{40}x^4 + e_{50}x^5 + \cdots .$$

The equations of the projection of C_u onto the plane $y = 0$ from the point $(0,0,1,0)$ are $y = 0$ and the second of equations (11). Instead of comparing these equations, however, we can get the desired information in a different manner. The expansions for C_u can be obtained from (8) by setting $\Delta v = 0$. The equations of the desired projection are $y = 0$ and the expansion obtained from (8) by eliminating Δu from the series for x and z, after Δv has been set equal to zero, thus obtaining z as a power series in x. The series (12) above will contain all the terms of this series plus a number of terms which were contributed by the expansion for y. The latter terms will represent the difference between the two series. If we examine the terms of series (12), we find that y contributes nothing to terms of degree less than six. Hence the two series will be identical to terms of degree five. Thus we can state the following theorem:

The section of the surface by the osculating plane of one of the curves of the parametric conjugate net at P_x has six-point contact with the projection of that curve from its ray point onto the osculating plane.

We investigate the possibility of seven-point contact in the same fashion. The coefficient e_{60} contains all the terms of the coefficient of the sixth degree term in the second of equations (11), and in addition, terms which were contributed by the positive powers of y. These latter terms are $N\mathfrak{N}^2/72r^2$ from y^2 and $L\mathfrak{N}(H - \mathfrak{N})/36r$ from x^3y. The condition for seven-point contact is that the sum of these terms be zero or

$$L\mathfrak{N}(2H - \mathfrak{N}) = 0.$$

Since L cannot be zero, and since $\mathcal{H} = 0$ is the condition that the u-curves be plane curves, we have the following theorem:

The section of the surface by the osculating plane of a curve of the parametric conjugate net at P_x has seven-point contact with the projection of the curve from its ray point onto that osculating plane if the parametric curve is a plane curve or if $\mathcal{H} = 2H$.

We next consider the projection of the curves of the conjugate net onto the tangent plane from an arbitrary point (a,b,c) not in the tangent plane of the surface at P_x. The coordinates of the point (X,Y,Z), which is the projection of the point (x,y,z), are given by

$$\begin{aligned} X &= (x + \lambda a)/(1 + \lambda), \\ Y &= (y + \lambda b)/(1 + \lambda), \\ Z &= (z + \lambda c)/(1 + \lambda) = 0. \end{aligned} \tag{13}$$

Hence we have $\lambda = -z/c$. Using (11) for the coordinates of a point on C_u, we get for the coordinates of the point of projection $Z = 0$ and the following expansions:

$$\begin{aligned} X &= x - aLx^2/2c + (3L - 8aL\mathfrak{C}')x^3/6c + L(32c\mathfrak{C}' - acb_4 \\ &\quad - 6aL)x^4/24c^2 + L(b_4c + 6L - 24ab_5 - 32aL\mathfrak{C}')x^5/24c^2 + \cdots, \\ Y &= -bLx^2/2c + (c\mathcal{H} - 8rbL\mathfrak{C}')x^3/6rc + (c^2\mathcal{H}[I + 16\mathfrak{C}'] \\ &\quad - cbrLb_4 - 6brL^2)x^4/24rc^2 + \cdots . \end{aligned}$$

Setting Y equal to a power series in X with undetermined coefficients and demanding that this equation be satisfied identically in x by the two series above we obtain

$$Y = -bLX^2/2c + (\mathcal{H}c^2 - 8\mathfrak{C}'bcrL - 3abrL^2)x^3/6r^2c + d_4x^4 + \cdots , \tag{14}$$

where

$$d_4 = aL(c^2\mathscr{H} - 16bcrL\,\mathfrak{C}' - 6rabL^2)/4c^2r + bL(3a^2L^2 + 12\,cL - 32acL\,\mathfrak{C}')/24c^3 + \mathscr{H}[I + 16\,\mathfrak{C}']/24r - bLb_4/24c - bL^2/4c^2.$$

Writing the general equation of a conic and demanding that the series (14) satisfy this equation identically in X to terms of the fourth degree we find as the equations of the osculating conic of the projection at P_x, z = 0 and

$$(15)\quad bLX^2/2c + (\mathscr{H}c^2 - 8Lbcr\,\mathfrak{C}' - 3abrL^2)XY/3Lbcr + CY^2 + Y = 0,$$

where

$$C = -4d_4c^2/L^2b^2 + 2(\mathscr{H}c^2 - 8bcrL\,\mathfrak{C}' - 3abrL^2)^2/9b^3L^3cr^2.$$

The polar line of point $P_{x_{-1}}$ with respect to this conic has equations Z = 0 and

$$(16)\qquad bLX/c + (\mathscr{H}c^2 - 8Nbc\,\mathfrak{C}' - 3NLab)Y/3Nbc = 0.$$

We note that any point on the line joining P_x and (a,b,c) has nonhomogeneous coordinates (ka,kb,kc). Since the coordinates a, b, and c enter the equation (14) homogeneously we can make the following statement:

The polar line of $P_{x_{-1}}$ (P_{x_1}) with respect to the osculating conic of the projection of C_u (C_v) from a point (a,b,c) onto the tangent plane at P_x is unchanged as the point (a,b,c) varies along a line χ_1.

By the same reasoning we see that all projections from a line χ_1 have four-point contact at P_x. In order to have five-point contact the terms of d_4 which are not homogeneous in a, b and c must be equal in all cases. Hence $b^2/2c^2 = 0$ and therefore b = 0. So in order to have five-point contact for all projections of C_u the point of projection must move along a line in the osculating plane of C_u at P_x.

The pole of the v-tangent, $X = Z = 0$, with respect to the osculating conic of the projection of C_u is the point

$$(\mathcal{H}c/3Nb - 8\mathcal{C}'/3 - aL/c, -1, 0, 0).$$

By symmetry, the pole of the u-tangent, $Y = Z = 0$ with respect to the osculating conic of the projection of C_v is

$$(\mathcal{K}c/3La - 8\mathcal{B}'/3 - bN/c, 0, -1, 0).$$

If we ask that the join of these two points be the principal join of the surface, whose equations are $Z = 0$ and $8(\mathcal{C}'X + \mathcal{B}'Y) = 3$, we find that this is possible only if $\mathcal{H} = \mathcal{K}$ and the point of projection in nonhomogeneous coordinates is

$$(\sqrt{\mathcal{H}c/3NL},\ \sqrt{\mathcal{K}c/3NL},\ c).$$

We next ask for conditions on the point of projection in order that the osculating conic of the projection of C_u at P_x shall have as its equation $Y = X^2$. Setting the coefficient of X^2 in (15) equal to minus one and the coefficients of XY and of Y^2 equal to zero we find

$$(\mathcal{H}c^2 - 8Lbcr\mathcal{C}' - 3abrL^2) = 0, \qquad d_4 = 0.$$

In this case the point of projection will have its coordinates a, b, c given by

$$a = -(\mathcal{H} + 16r\mathcal{C}')c/6N, \qquad b = -2c/L,$$
$$c = 72Lr^2/(64\mathcal{H}\mathcal{C}'r - \mathcal{H}^2 - 1280\mathcal{C}'^2r^2 + 6\mathcal{H}rI + 12r^2b_4).$$

The expansion (14) then takes the form

$$Y = X^2 + d_5X^5 + \cdots .$$

We next consider the special case arising when the point of projection is the point P_y. The equations $z = 0$ and the first of equations (11) represent the projection of C_u from P_y onto the tangent plane at P_x. In this case the osculating conic of the projection will have the equation $Y^2 = 0$. The eight point nodal cubic will have the equation $Y^3 = 0$. If we project from some other point on the axis having nonhomogeneous coordinates $(0,0,c)$ we find that the equations for the projection of C_u take the form

$$Z = 0,$$
$$Y = \mathcal{H}X^3/6r + \mathcal{H}(I + 16\,\mathfrak{C}')X^4 + (\mathcal{H}ca_5 - 20\mathcal{H}L)X^5/120rc + \cdots .$$

Since this expansion and the first of (11) are identical to terms of degree four, we can conclude that all projections from points on the axis have five-point contact at P_x. If we ask that all projections from points on the axis have six-point contact at P_x we find that $\mathcal{H}L/6rc$ must be zero. This is possible only if $\mathcal{H} = 0$ since $L \neq 0$ and the point $(0,0,c)$ is not the point P_y. Thus all projections of the curve C_u onto the tangent plane from a point on the axis, other than P_y, have six-point contact if and only if the u-curves are plane curves.

Now let us project the section of the surface by the osculating plane of C_u at P_x from the point (a,b,c) onto the tangent plane at P_x. Using the equations of projection (13) and the series (12), we find, by the method of undetermined coefficients, that the projection is represented by the equations $Z = 0$ and

$$(17) \qquad Y = -bLX^2/2c - bL(8c\,\mathfrak{C}' + 3La)X^3/6c^2 + c_4X^4 + \cdots ,$$

where

$$c_4 = -4bL\,\mathfrak{C}'^2/c - bL\,\mathfrak{C}'(\log \mathfrak{C}'\sqrt{r})_u + bL^2/4c^2 - 10abL^2\,\mathfrak{C}'/3c^2 - 5a^2bL^3/8c^3.$$

Comparing this expansion with equation (14) we find that they are identical to terms of second degree. Hence the projection of the curve C_u onto the tangent plane from a point (a,b,c) has three-point contact with the projection of the section of the surface by the osculating plane of C_u at P_x from the same point onto the tangent plane. If we examine the third degree terms we see that the two projections have four-point contact when $\mathcal{H} = 0$, or when the u-curves are plane curves.

The osculating conic of the curve (17) is found by the usual methods to have the equations $Z = 0$ and

$$bLX^2/2c - (8\mathcal{C}'c + 3aL)XY/3c - [2(8\mathcal{C}'c + 3aL)^2/27bc^2L + 4a_4c^2/b^2L]Y^2 + Y = 0.$$

The polar line of the point $P_{x_{-1}}$ with respect to this conic is the line

$$3bLX - (8\mathcal{C}'c + 3aL)Y = 0,$$
$$Z = 0.$$

By symmetry we can write the equations of the polar line of P_{x_1} with respect to the osculating conic of the projection onto the tangent plane of the section of the surface by the osculating plane of C_v at P_x; these equations are

$$(8\mathcal{B}'c + 3bN)X - 3aNY = 0,$$
$$Z = 0.$$

Since these equations are homogeneous in a, b, c we can conclude as before that these lines are unchanged when the point (a,b,c) varies along a line $\mathcal{L}_1$. If we choose the point of projection to be the point whose coordinates are given by

$$a = -4(\mathfrak{C}' + \mathfrak{B}')/3L, \qquad b = 4(\mathfrak{C}' - \mathfrak{B}')3N, \qquad c = 1,$$

the equations of the two lines above reduce to $z = 0$ and

$$LX - NY = 0, \qquad LX + NY = 0,$$

respectively. These lines separate the tangents of the conjugate net harmonically.

Next we shall consider the intersection of the surface and a variable plane through the axis. Let the equation of the plane be $y = \rho x$. The equations of the projection of this section from the point P_{x_1} onto the plane $y = 0$ are $y = 0$ and the equation obtained by eliminating y between $y = \rho x$ and the expansion (9). This latter equation is

$$\begin{aligned} z = {} & (L + \rho^2 N)x^2/2 + 4(L\mathfrak{C}' + N\rho^3\mathfrak{B}')x^3/3 + \Big\{L\mathfrak{C}'[12\mathfrak{C}' \\ & + (\log \mathfrak{C}'\sqrt{r})_u]/3 + \rho L(H - \mathcal{H})/6 + N\rho^3(K - \mathcal{K})/6 \\ & + N\mathfrak{B}'[12\mathfrak{B}' + (\log \mathfrak{B}'/\sqrt{r})_v]\rho^4/3\Big\}x^4 + \cdots . \end{aligned} \tag{18}$$

If we write the most general equation of the second degree in x and z and demand that the series (18) satisfy this equation identically to terms of x^4 we get, as the equations of the osculating conic of the projection, $y = 0$ and

$$\begin{aligned} & (L + \rho^2 N)x^2/2 + 8(L\mathfrak{C}' + N\mathfrak{B}'\rho^3)xz/3(L + N\rho^2) \\ & - \Big\{128(L\mathfrak{C}' + N\mathfrak{B}'\rho^3)^2 - 36(L + N\rho^2)(e_{40} + \rho e_{31} + \rho^3 e_{13} \\ & + \rho^4 e_{04})\Big\} z^2/9(L + N\rho^2)^3 - z = 0. \end{aligned} \tag{19}$$

The equations of the osculating conic of the section are $y = \rho x$ and (19). The locus of these conics as ρ varies is found, by eliminating ρ by means of the equation $y = \rho x$, to be a surface of degree eight, whose equation is

$$16(L\mathfrak{C}'x^3 + N\mathfrak{B}'y^3)[16(L\mathfrak{C}'x^3 + N\mathfrak{B}'y^3)z^2 - 3(Lx^2 + Ny^2)^2z]$$
$$+ 9(Lx^2 + Ny^2)[(Lx^2 + Ny^2)^2(2z - Lx^2 - Ny^2)$$
$$- 8z^2(e_{40}x^4 + e_{31}x^3y + e_{13}xy^3 + e_{04}y^4)] = 0 .$$

The polar line of $P_{x_{-1}}$ with respect to the conic (19) has equations $y = 0$ and

$$3(L + N\rho^2)x + 8(L\mathfrak{C}' + N\mathfrak{B}'\rho^3)z = 0. \tag{20}$$

This is the projection of a line, in the variable plane through the axis, whose equations are $y = \rho x$ and (20). The locus of this line as ρ varies is found by the elimination of ρ to be the quartic cone whose equation is

$$3(Lx^2 + Ny^2)^2 + 8(L\mathfrak{C}'x^3 + N\mathfrak{B}y^3)z = 0,$$

and having the axis as generator. The polar plane of $P_{x_{-1}}$ with respect to this cone is the plane

$$3Lx + 2\mathfrak{C}'z = 0,$$

and the polar plane of P_{x_1} is the plane

$$3Ny + 2\mathfrak{B}'z = 0.$$

The plane determined by the axis and the line of intersection of these two polar planes is the plane

$$\mathfrak{B}'x - r\mathfrak{C}'y = 0,$$

which is the canonical plane of the conjugate parametric net.[5]

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[5]Davis, op. cit., p. 18.

4. Analogues of the Quadrics of Lie. We shall now develop equations for a pair of quadric cones associated with the parametric curves through a point on the surface, which, due to their origin, may be regarded as analogues of the Quadrics of Lie. We consider a point x on a C_v and two neighboring points on the same curve. The tangents of the u-curves through these three points determine a quadric. We shall be interested in the equation of the limiting quadric as the two neighboring points independently approach P_x along the curve C_v. Let X be a point on C_v near P_x. The coordinates of P_X are then given by the Taylor series

$$X = x + x_v \Delta v + x_{vv} \Delta v^2/2 + \cdots .$$

The point X_u on the tangent of the u-curve through P_X has coordinates given by the expansion

$$X_u = x_u + x_{vu} \Delta v + x_{vvu} \Delta v^2/2 + \cdots .$$

The point Y, defined by

$$Y = hX + kX_u,$$

lies on the tangent of the u-curve through P_X. Replacing the derivatives of x by linear combinations of x, x_{-1}, x_1, y, we have

$$Y = y_1 x + y_2 x_{-1} + y_3 x_1 + y_4 y,$$

where the local coordinates of Y referred to the basic tetrahedron are given by

$$\begin{aligned}
y_1 &= h + kb + (ha + k[c + 2ab])\,\Delta v + \Big\{ h(q + \delta a) + k[(c + 2ab)_v \\
&\qquad + a(c + 2ab) + aK - bNm]\Big\} \Delta v^2/2 + \cdots , \\
y_2 &= k + ka\,\Delta v + k(a_v + a^2)\,\Delta v^2/2 + \cdots , \\
y_3 &= (h + kb)\,\Delta v + (\delta h + k[b_v + c + b\delta + ab])\,\Delta v^2/2 + \cdots , \\
y_4 &= N(h + kb)\,\Delta v^2/2 + \cdots .
\end{aligned} \tag{21}$$

Writing the general equation of a quadric surface in homogeneous coordinates and demanding that the series (21) satisfy this equation identically in h and k and identically in Δv to terms of the second degree, we obtain the equation

$$Nx_3^2 - c_1x_4^2 - 2x_1x_4 + c_2x_3x_4 = 0.$$

This equation represents a two parameter family of quadric cones with vertices at $P_{x_{-1}}$ and having the tangent of the u-curve at P_x as generator. Going through a similar procedure for three points on a u-curve we arrive at the equation

$$Lx_2^2 - \bar{c}_1x_4^2 - 2x_1x_4 + \bar{c}_2x_2x_4 = 0,$$

which represents a two-parameter family of cones with vertices at P_{x_1} and having the tangent of the v-curve at P_x as generator.

If we ask for the cones of each family which pass through P_y we find that for these $c_1 = \bar{c}_1 = 0$. If we further stipulate that the ray and axis shall be reciprocal polars we have the condition that $c_2 = \bar{c}_2 = 0$. Under these conditions we have two cones whose equations are respectively,

(22) $$Nx_3^2 - 2x_1x_4 = 0, \qquad Lx_2^2 - 2x_1x_4 = 0.$$

The equations of the projection of the intersection of these two cones from P_y onto the tangent plane of the surface at P_x are $x_4 = 0$ and the equation obtained by eliminating x_4 from the two equations above. Since the elimination of x_4 yields the equation

$$Lx_2^2 - Nx_3^2 = 0,$$

we see that the projection from P_y onto $x_4 = 0$ of the intersection of the cones analogous to the quadrics of Lie is a pair of lines which are the associate tangents at P_x.

5. Osculating quadric cones. We shall now develop the equations of the osculating quadric cones of the parametric curves through P_x, and investigate the relationship between these and the cones of the preceding section.

Writing the most general homogeneous equation of the second degree in x, y, z and asking that the expansions (11) for the u-curve satisfy this equation to terms of degree six, we get, as the equation of the osculating quadric cone of C_u at P_x,

$$N(36a_5 + 1280\mathfrak{C}'^2 - 960I\mathfrak{C}' - 45I^2 - 120b_4)y^2/80\mathfrak{N} - 3Lxy/2 + (3I - 16\mathfrak{C}')yz/4 + \mathfrak{N}z^2/N = 0.$$

We shall refer to this cone as the u-cone. By symmetrical substitutions we can write the equation of the osculating quadric cone of C_v at P_x, and shall refer to it as the v-cone. These cones will have their vertices at P_x, and will have the u-tangent and v-tangent respectively as generators. The polar plane of P_y with respect to the u-cone has the equation

$$N(3I - 16\mathfrak{C}')y + 8\mathfrak{N}z = 0, \tag{23}$$

and similarly the equation of the polar plane of the point P_y with respect to the v-cone is

$$L(3I' - 16\mathfrak{B}')x + 8\mathfrak{K}z = 0. \tag{24}$$

We find that the planes (23) and (24) are the polar planes of any point on the axis with respect to the u-cone and v-cone respectively.

To find the projection from the point P_y upon the tangent plane of the intersection of the osculating u-cone and the cone which is the analogue of the Quadric of Lie for the u-curve, we eliminate x_4, or z, between the two equations. This will show

that the projection of the intersection consists of the u-tangent $x_3 = 0$ and a cubic curve whose equation is

$$N(36a_5 + 1280\mathscr{C}'^2 - 960I\mathscr{C}' - 45I^2 - 120b_4)y/80\mathscr{H} - 3Lx/2 + N(3I - 16\mathscr{C}')y^2/8 + \mathscr{H}Ny^3/4 = 0.$$

It is easily verified that this cubic goes through the points P_x and $P_{x_{-1}}$, having a cusp at the latter point.

6. The ruled surfaces of rays. We next investigate the ruled surfaces of rays for the parametric curves through P_x. We shall develop a power series representation for the coordinates of a point on the ruled surface of rays for the u-curve through P_x, and use this series to obtain the equation of the osculating quadric of the ruled surface of rays. Then, by symmetric substitutions, we can write the equation of the osculating quadric of the ruled surface of rays for the v-curve through P_x. We shall then consider the intersection of these two quadrics and the polar planes of certain points with respect to the quadrics.

As the point P_x generates a u-curve on the surface S, the points $P_{x_{-1}}$ and P_{x_1} generate u-curves of the nets $N_{x_{-1}}$ and N_{x_1} respectively. A point X_1 near x_1 on the u-curve of the net N_{x_1} has coordinates given by the Taylor series,

$$X_1 = x_1 + x_{1u}\Delta u + x_{1uu}\Delta u^2/2 + \cdots . \tag{25}$$

Similarly a point X_{-1} near x_{-1} on the u-curve of the net $N_{x_{-1}}$ has coordinates

$$X_{-1} = x_{-1} + x_{-1u}\Delta u + x_{-1uu}\Delta u^2/2 + \cdots . \tag{26}$$

The derivatives of x_1 and x_{-1} appearing in the above expansions are given by the equations

$$x_{1u} = Hx + bx_1, \qquad x_{-1u} = Lmx + (\alpha - b)x_{-1} + Ly,$$
$$x_{1uu} = (H_u + 2Hb)x + Hx_{-1} + (b_u + b^2)x_1$$
$$x_{-1uu} = [(Lm)_u + Lm\alpha + LP]x + [2Lm + (\alpha - b)_u + (\alpha - b)^2]x_{-1} + Lsx_1 + L(\alpha - \gamma_u)y.$$

Substituting in (25) and (26) and collecting coefficients we have, writing terms of degree less than three,

$$X_1 = \{H\Delta u + (H_u + 2Hb)\Delta u^2/2 + \cdots\}x + (H\Delta u^2/2 + \cdots)x_{-1} + \{1 + b\Delta u + (b_u + b^2)\Delta u^2/2 + \cdots\}x_1 + (\cdots)y,$$
$$X_{-1} = \{Lm\Delta u + [(Lm)_u + Lm\alpha + LP]\Delta u^2/2 + \cdots\}x + \{1 + (\alpha - b)\Delta u + [2Lm + (\alpha - b)_u + (\alpha - b)^2]\Delta u^2/2 + \cdots\}x_{-1} + \{Ls\Delta u^2/2 + \cdots\}x_1 + \{L\Delta u + L(\alpha - \gamma_u)\Delta u^2/2 + \cdots\}y.$$

The point Z defined by the equation

$$Z = hX_1 + kX_{-1}$$

lies on the generator of the ruled surface of rays for the u-curve through P_x. Forming this combination and collecting coefficients we have

$$Z = \{(hH + kLm)\Delta u + [h(H_u + 2Hb) + k\{LP + Lm\alpha + (Lm)_u\}]\Delta u^2/2 + \cdots\}x + \{k + k(\alpha - b)\Delta u + \{hH + k(2Lm + [\alpha - b]_u + [\alpha - b]^2)\}\Delta u^2/2 + \cdots\}x_{-1} + \{h + hb\Delta u + [h(b_u + b^2) + kLs]\Delta u^2/2 + \cdots\}x_1 + \{kL\Delta u + kL(\alpha - \gamma_u)\Delta u^2/2 + \cdots\}y.$$

Thus the local coordinates of the point Z are given by

$$z_1 = (hH + kLm)\Delta u + [h(H_u + 2Hb) + k\{LP + Lm + (Lm)_u\}]\Delta u^2/2 + \cdots .$$

$$
\begin{aligned}
z_2 &= k + k(\alpha - b)\Delta u + [hH + k(2Lm + \{\alpha - b\}_u \\
&\qquad + \{\alpha - b\}^2)]\Delta u^2/2 + \cdots , \\
z_3 &= h + hb\,\Delta u + [h(b_u + b^2) + kLs]\Delta u^2/2 + \cdots , \\
z_4 &= kL\,\Delta u + kL(\alpha - \gamma_u)\Delta u^2/2 + \cdots .
\end{aligned} \tag{27}
$$

If we write the equation of a general quadric and ask that it be satisfied by the expansions (27) identically in h and k and identically in Δu to terms of the second degree, we get as the equation of the osculating quadric of the ruled surface of rays for the u-curve through P_x,

$$
\begin{aligned}
2LH^2x_3x_4 &= 2HL^2x_1x_2 + L[H(2b - \alpha - \gamma_u) - H_u]x_1x_4 \\
&\quad - 2HL^2mx_2x_4 - [HLP + (Lm)_uH - LH_um \\
&\quad + HLm(2b - \alpha)]x_4^2 .
\end{aligned}
$$

By employing several of the invariants (4), (5), and (6), we can write this equation in the form

$$
\begin{aligned}
4H^2Lx_3x_4 &= 4HLx_1x_2 + [(8\mathfrak{C}' - \gamma_u)H - 2H_u]Lx_1x_4 - 2HL\mathfrak{D}x_2x_4 \\
&\quad - [2HLP + 2H\mathfrak{D}_u - \mathfrak{D}H_u + \mathfrak{D}H(4\mathfrak{C}' + \gamma_u/2)]x_4^2 .
\end{aligned} \tag{28}
$$

By symmetry we can write the equation of the osculating quadric of the ruled surface of rays for the v-curve through P_x; this equation is

$$
\begin{aligned}
4K^2Nx_2x_4 &= 4KNx_1x_3 + [(8\mathfrak{B}' + \gamma_v)K - 2K_v]Nx_1x_4 + 2KNr\mathfrak{D}x_3x_4 \\
&\quad - [2KNQ - 2K(r\mathfrak{D})_v + r\mathfrak{D}K_v - \mathfrak{D}rK(4\mathfrak{B}' - \gamma_v/2)]x_4^2 .
\end{aligned} \tag{29}
$$

We shall, for the sake of brevity, refer to these quadrics as Q_u and Q_v respectively.

The equation of the polar plane of P_{x-1} with respect to the quadric Q_u is

$$2Lx_1 - \mathfrak{D}x_4 = 0,$$

and the polar plane of P_{x_1} with respect to Q_v is represented by the equation

$$2Nx_1 + r\mathfrak{D}x_4 = 0.$$

It is easily shown that these two planes pass through the ray and separate the coordinate planes $x_1 = 0$ and $x_4 = 0$ harmonically. The polar plane of $P_{x_{-1}}$ with respect to the quadric Q_v is the tangent plane of the surface S at P_x, as is the polar plane of the point P_{x_1} with respect to the quadric Q_u.

The intersection of the quadrics Q_u and Q_v is of some interest. The equation of the projection of this intersection from P_x onto the plane $x_1 = 0$, other than $x_1 = 0$, is obtained by eliminating x_1 from the two equations (28) and (29). Performing this elimination leads to the equation

$$\begin{aligned} x_4&\left[2Hx_3 + \mathfrak{D}x_2 + \left\{P + \mathfrak{D}_u/L - \mathfrak{D}H_u/HL + \mathfrak{D}(4\mathfrak{C}' + \mathcal{I}_u/2)/2L\right\}x_4\right] \\ &\cdot\left[2Nx_3 + (4\mathfrak{B}' + \mathcal{I}_v/2 - K_v/K)x_4\right] = \\ x_4&\left[2Kx_2 - r\mathfrak{D}x_3 + \left\{Q - (r\mathfrak{D})_v/N + \mathfrak{D}K_v/NK - r\mathfrak{D}(4\mathfrak{B}'\right.\right. \\ &\left.\left. - \mathcal{I}_v/2)/2N\right\}x_4\right]\left[2Lx_2 + (4\mathfrak{C}' - \mathcal{I}_u/2 - H_u/H)x_4\right]. \end{aligned}$$

Factoring out x_4, we see that the ray is part of the intersection, and have left a conic in the plane $x_1 = 0$ which intersects the ray $x_1 = x_4 = 0$ in two points whose coordinates satisfy the relation

$$NHx_3^2 + 2LmNx_2x_3 - LKx_2^2 = 0.$$

These two points are the focal points of the ray[6], hence the following theorem is proved:

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[6]Lane, Projective Differential Geometry, p. 141.

The quadrics Q_u and Q_v intersect in the ray of the point P_x and in a cubic curve which intersects the ray in the foci of the ray.

VITA

Jerome Michael Sachs was born in Chicago, October 2, 1914. His early education was obtained in the public schools of Chicago. He received the degree of Bachelor of Science from the University of Chicago in 1936 and the degree of Master of Science from the same institution in 1937.

During the year 1936-37 he taught mathematics at Englewood Evening High School, and the following two years he served as part time instructor at Armour Institute of Technology. Since 1938 he has been an instructor at Wilson Junior College.

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